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Discrepencies of Theory as Against Empirical Results of Critical Speeds of Circular Cross Sectioned Shafting

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Title- Discrepencies of Theory as Against Empirical Results of Critical Speeds of Circular Cross Sectioned Shafting.

Object- The scope of this thesis included computations and tests showing the first critical speeds of two shafts each $\frac{5''}{8}$ dia.X 38" long one of C.R.S. and one of hard drawn brass. The purpose was to find and tabulate the discrepencies between the computed values for the critical speeds and the empirical values for critical speeds.

Theory- Any shaft accurately balanced; and either loaded, or unloaded, if run at a gardually increasing speed will reach a speed where it will vibrate considerably in the part that is unconstrained. At a slight further increase of speed this vibration will cut out and the system will run to higher speeds very smoothly. This speed where the vibration Amplitude is at maximum is known as the first critical speed of the system.

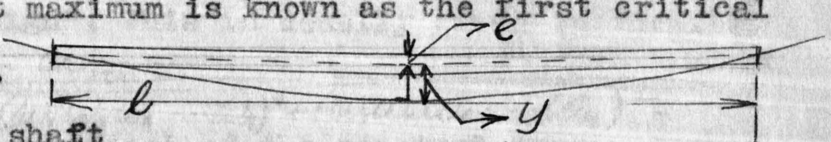


Fig. 1.

Consider first a shaft of known weight-w-,and known length-l-,as in Fig.1. No matter how closely this system is balanced it will be out of balance by a very small eccentricity as-e-,in Fig.1. Therefore, as soon as it begins to rotate it will deflect then a distance-y- Fig.1. creating an elastic line, whose total deflection is $y+e$. The load force causing deflection y, is the well known centrifugal force $M\omega^2 r = M\omega^2(y+e)$ for this case, The deflection of any beam may always be thrown into the form $P = K y_{max}$ When P=the external load and K is a constant which for this case is $\frac{384EI}{5l^3}$ from the general deflection formula "Strength of Materials". When E is the modulus of elasticity, and I is the cross

section moment of inertia. The general formula then becomes,
$P = M\omega^2(y+e) = K y_{max}$ or $y_{max} = \frac{M\omega^2 e}{K - M\omega^2}$
Inspection of this equation shows that when $K = M\omega^2$ then y becomes very large or $= \infty$. therefore solving for $\omega = \sqrt{\frac{K}{M}}$ gives the general equation for the first critical speed of the shaft in radians per second. This formula may be applied to any rotating member such as a disc mounted at the center of the shaft. For this case $\omega_2 = \sqrt{\frac{K_2}{M_2}}$ where $K_2 = \frac{48EI}{l^3}$ and $M_2 = \frac{w}{386}$ where w is the weight of the disc and 386=12g, acceleration constant g changed to inches per second^2. The combined critical speed of the shaft and the disc load at the center may be found by the use of Dunkerley's formula-pp.1111-Vol.2 Stodola's,"Steam and Gas Turbines". Also;-pp.465-Vol.1 Stodola, and pp.602-Normans,"Machine Design". This is usually given in the form

$$\omega = \frac{\omega_1 \omega_2 \omega_3 \cdots \omega_n}{\sqrt{(\omega_2\omega_3\cdots\omega_n)^2 + (\omega_1\omega_3\omega_4\cdots\omega_n)^2 + (\omega_1\omega_2\omega_4\cdots\omega_n)^2 \cdots}}$$

where ω is the critical speed of the combined system consisting of the separately computed critical speeds of the mass of the shaft alone, each disc mass added called ω_1, ω_2, etc. Dunkerley's original formula was
$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} \cdots + \frac{1}{\omega_n^2}$ pp.465 Vol.1. Stodola,"Steam and Gas Turbines".

List of apparatus used in the tests was as follows:

1. Cold Rolled Steel Shaft $\frac{5}{8}'' \times 38''$
2. Cold Rolled Brass Shaft $\frac{5}{8}'' \times 38''$
3. Steel disc weighing 3.50#.
4. Spring mounted brass stylus in disc $/\frac{1}{16}''$ off center of shaft, also very light spring brass stylus to fit unloaded shaft.
5. Cards 3x5" glazed with zinc sulphate with hole in center $/\frac{3}{8}''$ dia. for shaft to pass thru.
6. Speed lathe driven by General Electric continuous current shunt wound motor, TypeC.R.,Volts-220, Amps 2.3, Speed 0.0 to 1800.
7. Controller-- Cutler Hammer, armature resistance for starting duty, field resistance for regulating duty.
8. Hasler speed counter; range 0.0 to 10,000 R.P.M.
9. Foxboro Tachometer; range 0.0 to 20,000 R.P.M.
10. Riehlé tension testing machine.
11. Berry strain gage.
12. Special card holder mounted on lathe bed ways to facilitate bringing zinc coated cards item (5) against brass stylus on rotating shaft, item (4).

Fig. 4.

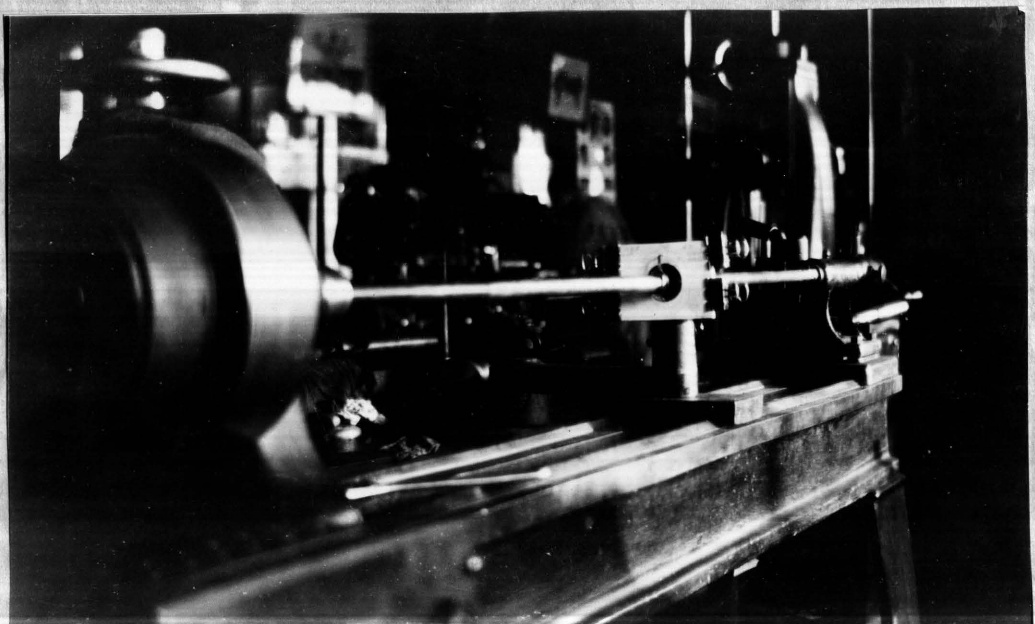


Fig. 5.

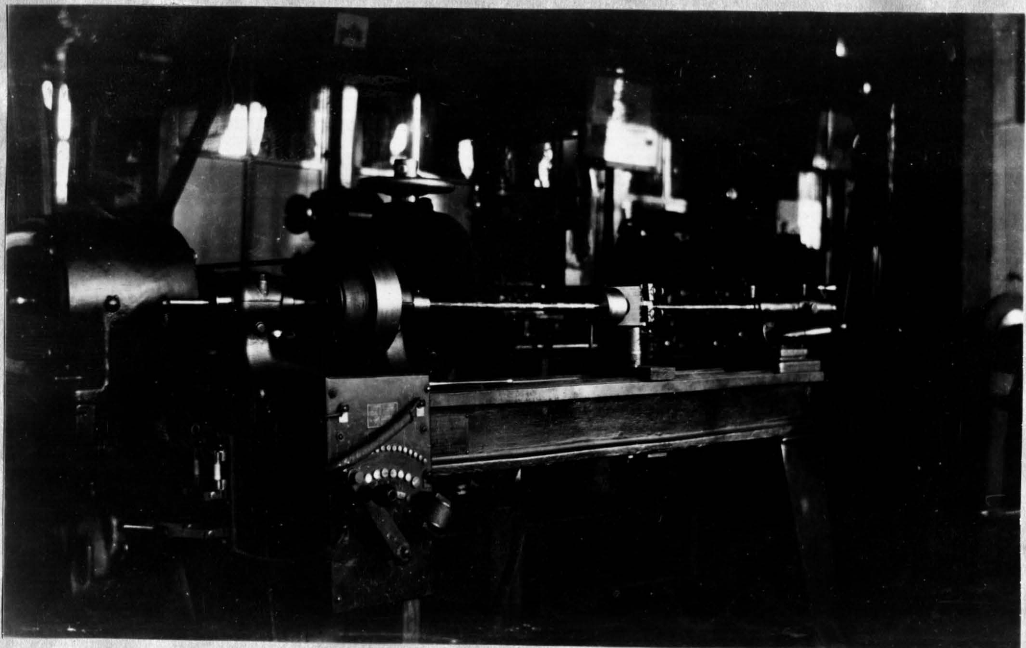


Fig. 6.

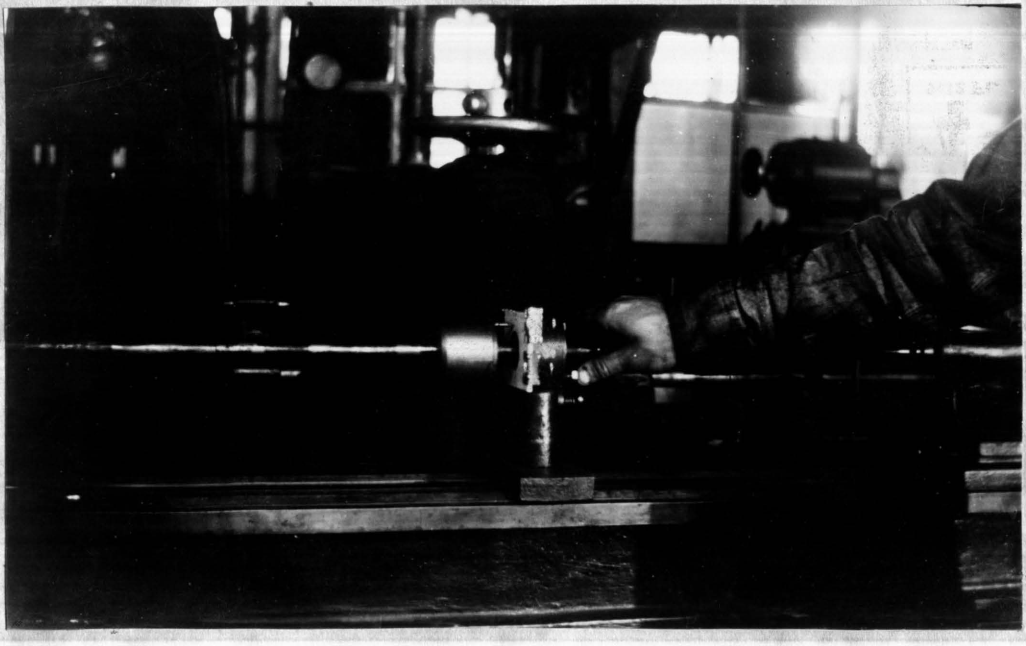


Fig. 7.

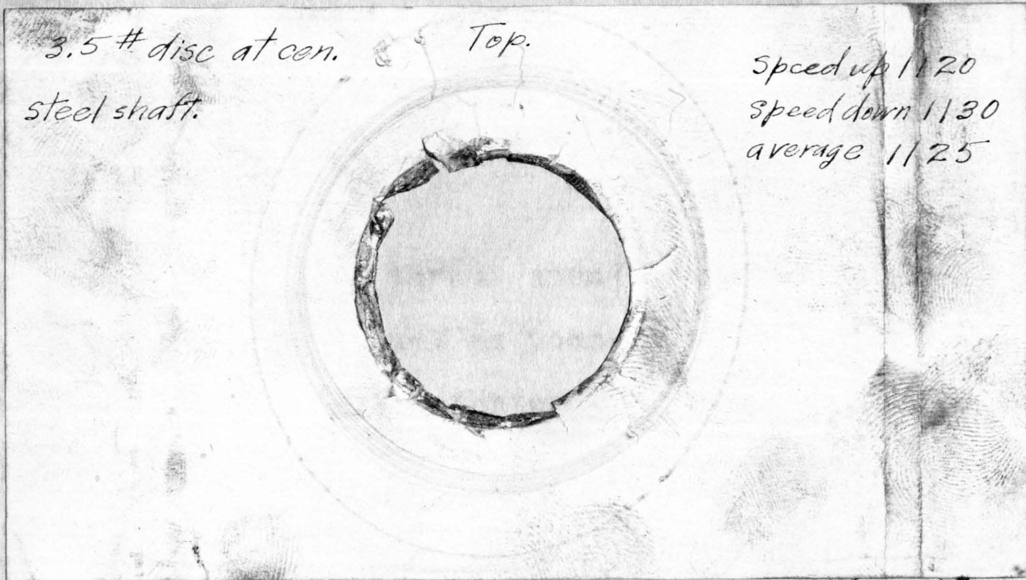

3.5 # disc at cen.
steel shaft.
Top.
speed up 1120
speed down 1130
average 1125

Fig. 8.

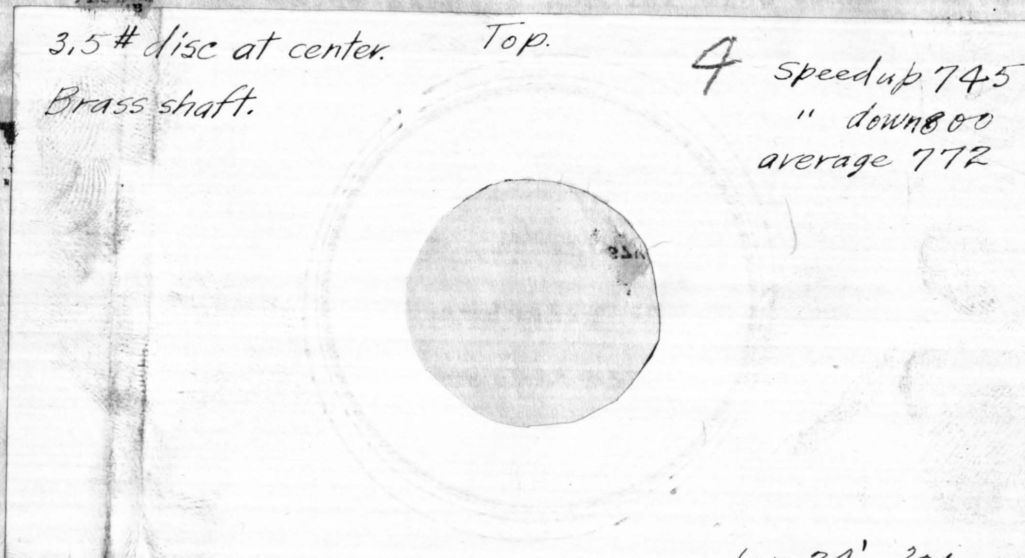

3.5 # disc at center.
Brass shaft.
Top.
4
speed up 745
" down 800
average 772
Jan, 24' - '31.

Fig. 9.

Procedure:-

Both shafts mentioned above were perpared by centering with a no. F center reamer, made by the Cleveland Twist Drill Co. New centers were made for the speed lathe, item (6)*, to fit not only the 60° angle but also the pilot hole as in Fig.3, so as to elliminate end thrust upon the shaft in action as much as possible. Both shafts were straightened and dynamically balanced.

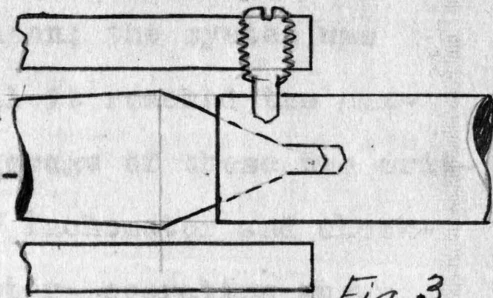


Fig. 3.

The special safety guard and sliding card holder, item (12), was constructed and mounted on lathe as shown in Fig. 5,6,7, etc., diagram sheet pp.5 a&b of this report.

A special brass wire spring stylus was made, having very small mass, and mounted upon each unloaded shaft as shown in Fig.4&5 pp.5a Also; a special disc mass of 3.50# weight and containing the spring loaded straight brass stylus at the side of disc facing the sliding card holder as in Fig.7 pp. 5b.

Each shaft both loaded and unloaded at center with the 3.50# disc were brot gradually up to the critical speed and at the instant the vibration appeared to be at a maximum the sliding card holder bearing the zinc sulphate coated card was brought for a small time against the elastic stylus drawing a figure of the exact motion of the stressed element of the shaft to a magnified scale ratio $\frac{17}{5}$. Because the radius of stylus was $1\frac{1}{16}$ and of extreme fiber of shaft was $\frac{5}{16}$. Fig.7 pp.5b.

* Items- here refer to List of apparatus.

It is well to note that Lissajous' figures gave only the relative motion of the center of the shaft which is not stressed at all.

In each test the shaft was run over the first critical speed coming to stability again with no vibration due to $+y$, changing in sign to $-y$. Thereupon; the system was gradually decelerated in speed until it reached the maximum of vibration amplitude. The average of these two critical speeds, as read on the Hassler Tachometer and checked by the Foxboro Dead Beat Tachometer- operation shown in Fig.6 pp. 5b, was recorded as the true critical speed of the shaft.

The Modulus of Elasticity E, used in the computation was first found by the usual method of pulling in a Riehle testing machine then the strain, $\triangle$, with the Berry Strain Gage, but the computed results for brass went so far awry that they were mistrusted; therefore, another method of getting E was used. Viz., a weight of 10# was hung at center of brass shaft as in Fig. 4 pp. 5b. A Hays Indicator was clamped to the card holder with the ram resting on top of the shaft, then by raising the weight clear of the shaft, and then allowing the shaft to carry the 10# weight the actual deflection y was obtained and E was computed as shown under computations pp.9.

All computations for critical speeds as well as for E; the modulus of elasticity for brass were shown under computations pp.9,10,& 11.

All positive results in these computations were emphasized by underscoring as, 1100 R.P.M., and the empirical result from test was recorded directly under these results.

The figures obtained by this method of getting the exact motion of the stressed element of shafts gave a surprising result as, unlike the Lissajous' figures, they appeared to be true ellipses in all cases. Therefore a good representative card was taken from each of the four types of loadings and tested by superimposing a true ellipse drawn on tracing paper over each of these representative cards. Fig.8 & 9 pp. 5c.

Computations

Computation for E for brass shaft.

$$E = \frac{wl^3}{48Iy} = \frac{10 \times \overline{38}^3}{48 \times .0075 \times .120} = 12,700,000.$$

where $w = 10^{\#}$, $l = 38''$, $I = \frac{\pi d^4}{64} = .0075$,

and $y = .120'' =$ reading of Hays indicator.

Computation for steel shaft.

Formula from theory, $\omega = \sqrt{\frac{K}{M}}$ = rad/sec.

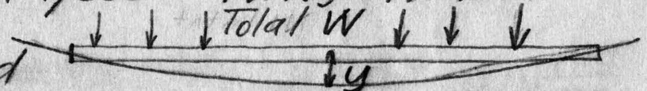
K = constant in deflection formula ($y = KP$)

y = deflection in inches, and P = load on shaft.

M = mass of the shaft or disc on shaft in

total pounds ÷ 386 = in/sec^2 = 12 × g = 12 × 32.2

Steel shaft unloaded



$$y_{max} = \frac{5wl^3}{384EI}$$

$K = \frac{384EI}{5l^3}$, $E = 29,500,000$ from test

on Riehlé machine. $I = \frac{\pi d^4}{64} = \frac{\pi \overline{.625}^4}{64} = .0075$

$$\therefore K = \frac{384 \times 29500000 \times .0075}{5 \times 38 \times 38 \times 38} = 310$$

$M = \frac{3.25^{\#}}{386} = .0084 \quad \therefore \sqrt{\frac{K}{M}} = \sqrt{\frac{310}{.0084}} = \sqrt{37000} =$

192 rad per sec. constant to change

rad per sec to R.P.M. $= \frac{60}{2\pi} = 9.55$

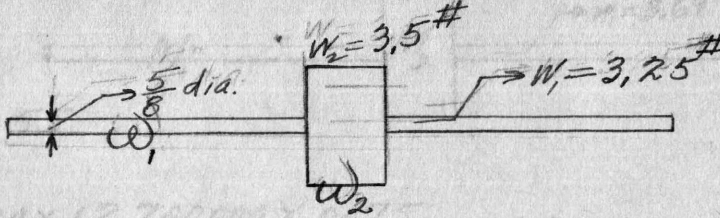
$\therefore 192 \times 9.55 =$ 1835 R.P.M = computed speed.

Actual critical speed = 1835 R.P.M. = average

of readings on machine.

Computations

Steel shaft loaded with 3.5# disc at center.



$$\omega_2 = \sqrt{\frac{K_2}{M_2}}$$

$$K_2 = \frac{48 \times 29\,500{,}000 \times .0075}{38 \times 38 \times 38} = 193$$

$$M_2 = \frac{3.5}{386} = .0091$$

$$\therefore \omega_2 = \sqrt{\frac{193}{.0091}} = \sqrt{21200} = 145 \frac{\text{rad}}{\text{sec.}}$$

ω = speed of whole system = shaft + disc =

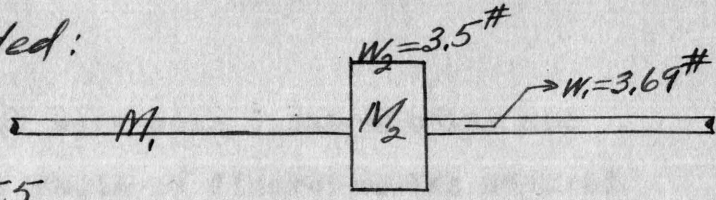
$$\omega = \frac{\omega_2 \omega_1}{\sqrt{\omega_2^2 + \omega_1^2}} = \frac{145 \times 192}{\sqrt{145^2 + 192^2}} = \frac{28000}{241} = 116 \frac{\text{rad}}{\text{sec.}}$$

$116 \times 9.55 =$ 1110 R.P.M.

Actual average observed critical speed from 10 readings of the Hassler tachometer = 1125 R.P.M.

Computations.

Brass shaft unloaded:



$$M_1 = \frac{3.69^{\#}}{386} = .00955$$

$$K_1 = \frac{384 EI}{5 \ell^3} = \frac{384 \times 12700000 \times .0075}{5 \times 38 \times 38 \times 38} = 133$$

$$\therefore \omega_1 = \sqrt{\frac{K_1}{M_1}} = \sqrt{\frac{133}{.00955}} = \sqrt{1'39{,}00} = 118 \text{ rad/sec}$$

$$118 \times 9.55 = \underline{1130 \text{ R.P.M.}}$$

Observed speed $\underline{1135 \text{ R.P.M.}}$

Brass shaft loaded:

$$M_2 = \frac{3.5}{386} = .0091$$

$$K_2 = \frac{48 \times 12700000 \times .0075}{38 \times 38 \times 38} = 83$$

$$\omega_2 = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{83}{.0091}} = \sqrt{9125} = 95.6 \text{ rad/sec.}$$

Combined system:

$$\omega_{brass} = \frac{\omega_2 \omega_1}{\sqrt{\omega_2^2 + \omega_1^2}} = \frac{\overset{11270}{95.6 \times 118}}{\sqrt{95.6^2 + 118^2}} = 74.2 \text{ rad/sec.}$$

$$74.2 \times 9.55 = \underline{709 \text{ R.P.M.}}$$

Actual average observed speed $\Big\}$ $\underline{770 \text{ R.P.M.}}$

Discussion of Results.

The results of this test were both disappointing and surprising because in these tests no discrepencies occured at all in the order recorded by Stodola pp.468. Vol.1. Taking the liberty to quote here:"An examination of these figures discloses that the observed critical speeds are through-out smaller than the theoretical, while the ratios of the speeds of the several orders agree reasonably with the theoretical values. The reason of the first deviation may be in the sympathetic vibrations of the very light and unsubstantial foundation and in the imperfect fixed bearings used in my experiments. The full explanation of this discrepency will be left to further experiments."

In this test the observed critical speeds for the steel shaft are larger than the theoretical but agree reasonably close. For the brass shaft they fall in the same order with a discrepency tending toward the empirical values being higher than the theoretical. No sympathetic vibrations could have been set up in this test as the lathe used was heavy and the bearings were well checked with indicators to make sure of rigid bearings.

Stodolas' results seem to point to a slipping tachometer draive which might easily have been true. It has been the writers experience that old tachometers tend to read low due to worn and dry parts in the instrument itself. The readings recorded in this report pp. 9-10 & 11 were taken with a specially constructed rubber block drive at rear of lathe spindle with a new Hassler Tachometer; also, these readings have nearly all been rechecked with a Foxboro Dead Beat Tachometer. The discrepency tending toward a higher actual critical speed in all cases than the computed values can be explained by the fact that the line of action of the eccentricity-e- Fig.1.- of the unloaded shaft does not lie in a plane with -e- of the disc at center of shaft, therefore e in the equation $P=M\omega^2(y+e)=Ky$ is not the algebraic sum of $e+e'$ but is really the vectorial sum of $e+e'$, in the combined rotating system. Since vectorial $e+e' > e+e'$ taken algebracially -y- becomes slightly neutralized therefore the actual critical speed would go higher than the theoretical.

By superimposing true ellipses over the figures having the same major and minor axes as the figures, the curves drawn by the vibrating shafts upon the zinc sulphate coated cards proved to be the true ellipse. These curves represented the actual motion of a stressed element of the shaft at the extreme fibre whereas, the Lissajous' figures showed only the actual motion of the center element of the shaft which would not be stressed at all. No odd or irregular figures like the Lissajous' figures were obtained at all in these tests.

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